Math 5707 Spring 2023

Matching theory Snippet 3: Max weight bipartite matching (Bondy-Murty §5.5, Schrijver §3.5)

Given $G = (X \sqcup Y, E)$ bipartite

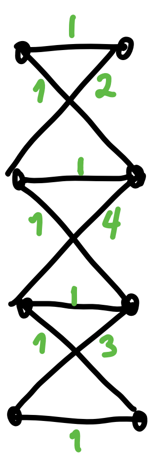
and edge weights $w: E \to \mathbb{R}_{\geq 0}$,

want to find a matching $M \subset E$

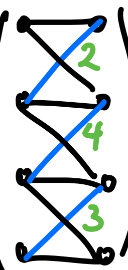
that maximizes total weight $w(M) := \sum_{e \in M} w(e)$

It will not always have max size $|M| = \nu(G)$!

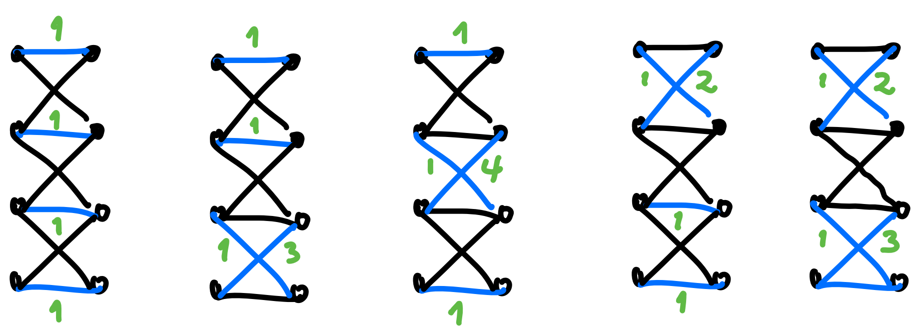
EXAMPLE



G, w

has $w\Bigg($  $\Bigg) = 2+4+3 = 9$ with 3 edges

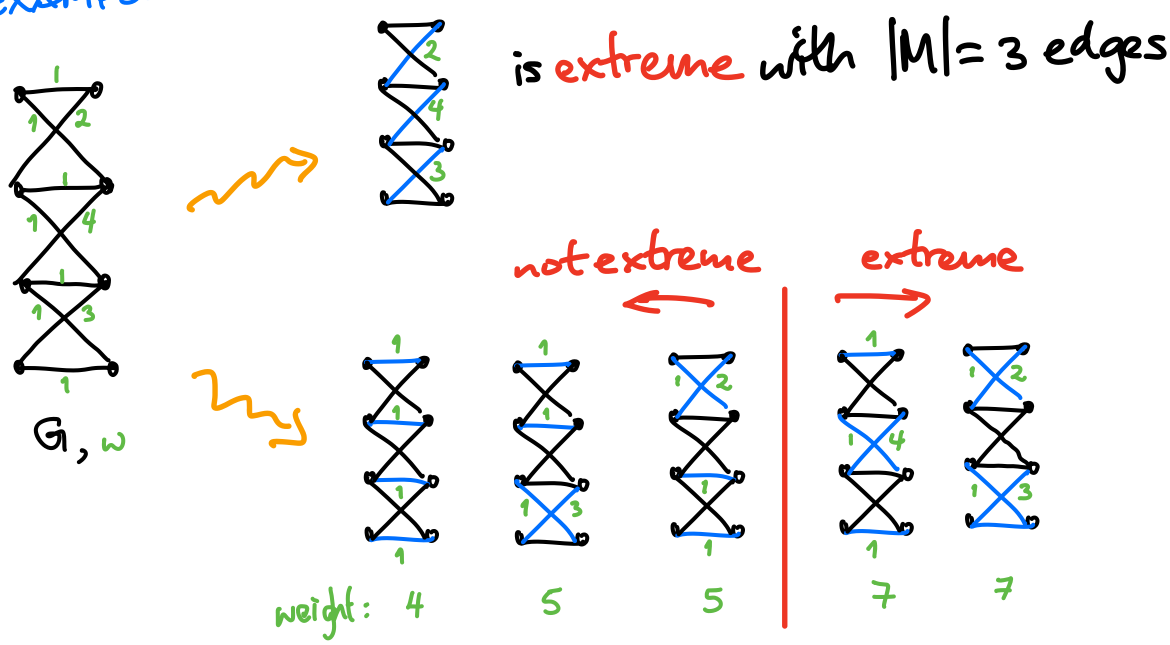
beating all of the matchings of size $4 = \nu(G)$:



weight: 4 5 7 5 7

DEFINITION: Call a matching $M \subset E$ **extreme** if it has max weight $w(M)$ among all matchings in G of the same size.

EXAMPLE



Kuhn 1955 gave a generalization of the Hungarian algorithm to find **one extreme matching M_i of each size** $|M_i| = i$ for $i = 0, 1, 2, \ldots, \nu(M)$:

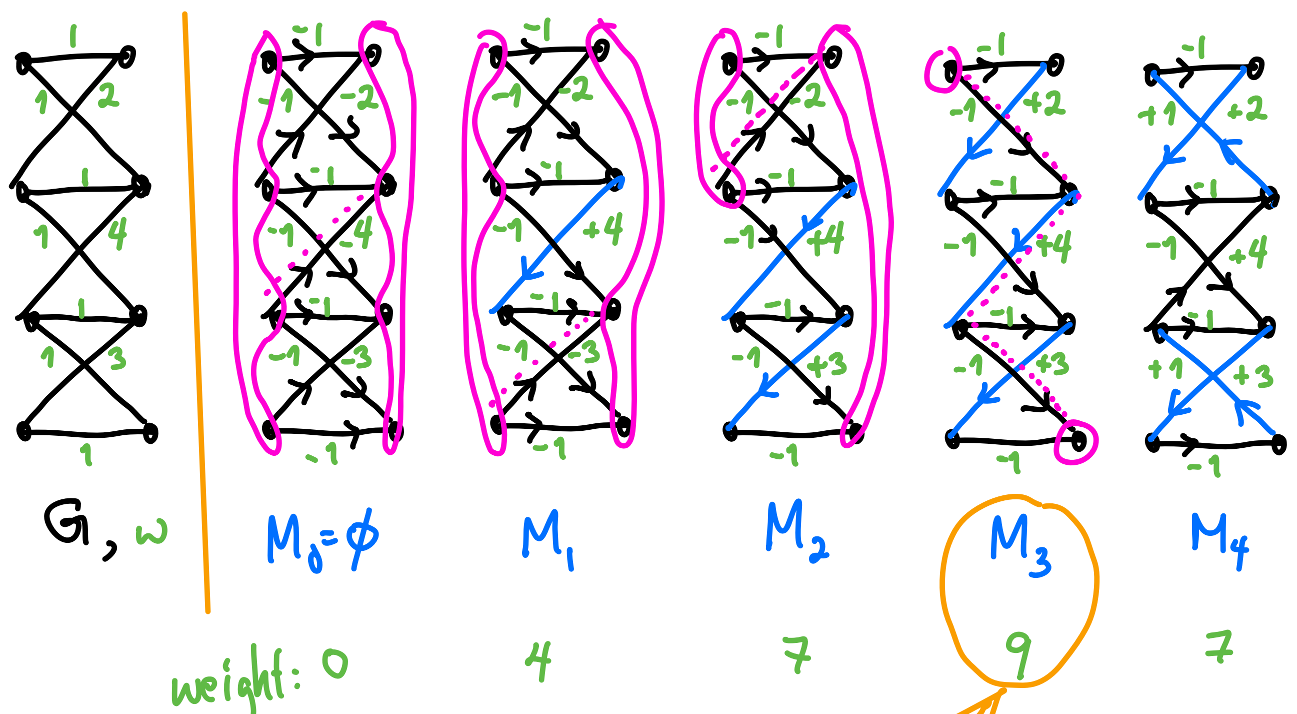
- Direct G via M_i to obtain digraph D_i as before:
  $x \rightarrow y$ not in M_i
  $x \leftarrow y$ in M_i
- Now put "lengths" on the arcs of D_i: $\begin{cases} x \xrightarrow{-w(e)} y & \text{not in } M \\ x \xleftarrow{+w(e)} y & \text{in } M \end{cases}$

- Search for directed paths P in D_i from any unmatched x in X to any unmatched y in Y, but of minimum ~~total~~ length among all such paths.

- If none exist, $|M_i| = \nu(M)$, so stop.
  If one exists, augment M_i along P to obtain M_{i+1}.

EXAMPLE

M-unmatched vertices circled



max weight matching is M_3

Why does it work as claimed?

PROPOSITION: If M_i was extreme, then so is M_{i+1}.

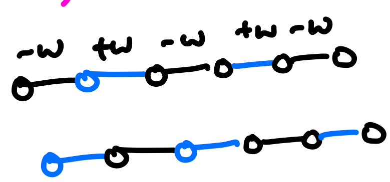
proof: For $i=0$, $M_0 = \emptyset$ is extreme.

Inductively, let M'_{i+1} be any extreme matching with $i+1$ edges. Want to show that

$$w(M_{i+1}) \geq w(M'_{i+1}).$$

We augmented M_i along a path P_i to obtain M_{i+1}.

Note $w(M_{i+1}) = w(M_i) - \ell(P_i)$



We know that $M_i \sqcup M'_{i+1}$ contains a connected component which is an M_i-augmenting path P'_i; use to create an M'_i matching with i edges such that M'_i augmented along P'_i gives M'_{i+1}.

Note again $w(M'_{i+1}) = w(M'_i) - \ell(P'_i)$

By construction in Kuhn's algorithm,

$$\ell(P'_i) \geq \ell(P_i)$$

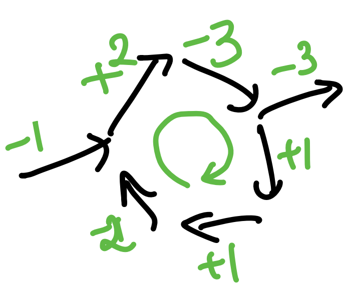
Hence

$$w(M_{i+1}') = w(M_i') - \ell(P_i')$$
$$\leq w(M_i') - \ell(P_i)$$
$$\leq w(M_i) - \ell(P_i)$$
$$= w(M_{i+1}) \qquad \blacksquare$$

since M_i is extreme (among matching with i edges)

Remaining issue:

Can one quickly find directed paths P of minimum length in the digraph D_i when there are negative edge lengths present ?!

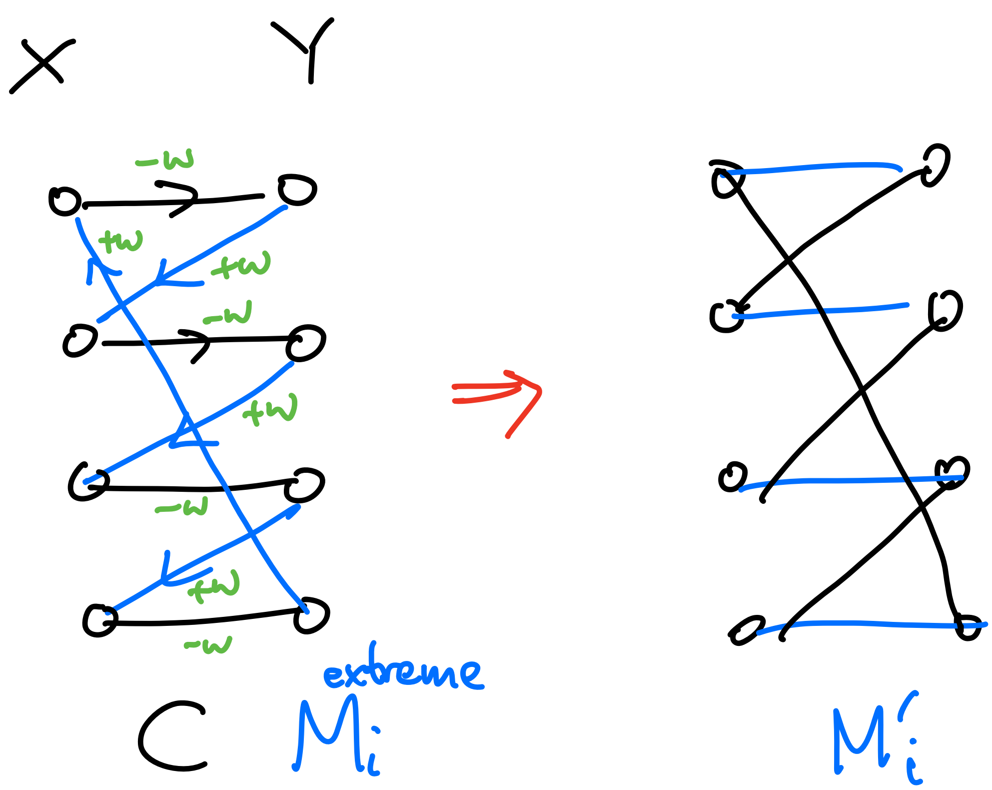


A problem: cycle C with $\ell(C) < 0$!

If there are no such directed cycles C with total length $\ell(C) < 0$, then $\exists$ an obvious algorithm (Bellman-Ford; Schrijver §1.3) to find min length directed paths from $x \to \to \to \to \to \to y$.

LEMMA: For the digraphs D_i based on M_i in Kuhn's algorithm, there are no cycles C with $\ell(C) < 0$.

proof: If we had such a cycle C, it would like this:



$$w(M_i') = w(M_i) - \ell(C)$$

$$> w(M_i)$$

Contradiction. $\blacksquare$